1. Use the principle of mathematical induction to show that $1+2+3+\cdots+n=\frac{n(n+1)}{2}$ for all $n\in\mathbb{N}$. [1 point]

Sol: ① check for $n=1$: $1=\frac{1(1+1)}{2}$ ✓ base (0.1 p.)

② assume that the statement holds for $k\in\mathbb{N}$, $k\geq 1$, that is:

$$1+2+\cdots+k=\frac{k(k+1)}{2}$$

We now check if the statement is true for $k+1$:

$$1+2+\cdots+k+(k+1)=\frac{k(k+1)}{2}+k+1$$

$$=\frac{k(k+1)+2(k+1)}{2}=\frac{(k+1)(k+2)}{2}$$ base + induction (0.75 p.)

This shows, by induction, that the statement is valid for $k+1$.

Thus, by mathematical induction, we have shown that $\sum_{i=1}^{n} i=\frac{n(n+1)}{2}$

all steps correct + conclusion (1p)

2 (v1). Use mathematical induction to prove that $3^{2n}-1$ is divisible by 8 $\forall\, n \in \mathbb{N}$. [1p.]

Sol: ① Check for $n=1$: $3^2-1=8$, which is divisible by 8 ✓ base (0.1p)

② Assume the statement is true for $k \in \mathbb{N}$. That is: $3^{2k}-1=8A$ where A is a positive integer. We now check if the statement holds for $k+1$:

$$3^{2(k+1)}-1 = 3^{2k+2}-1 = 9\cdot 3^{2k}-1+9-9 = 9\left[3^{2k}-1\right]+8 = 9(8A)+8.$$

which is divisible by 8. ✓ base+ induction (0.75p)

So, by mathematical induction we have shown that $3^{2n}-1$ is divisible by 8 for all $n \in \mathbb{N}$.

all steps correct + conclusion (1p)

2 (v2) Use mathematical induction to prove that $4^n+15n-1$ is divisible by 9 for all $n \in \mathbb{N}$.

Sol: ① Check for $n=1$: $4^1+15-1=18$ which is divisible by 9. ✓ base (0.1p)

② Assume the statement holds for $k \in \mathbb{N}$. That is $4^k+15k-1$ is divisible by 9, which can be rewritten as $4^k+15k-1=9A$, where A is an integer. We now check if the statement holds for $k+1$:

$$4^{k+1}+15(k+1)-1 = 4\cdot 4^k+15k+15-1 = 4\left(4^k+15k-1\right)-45k+18 =$$

$$4(9A)+9\left[-5k+2\right],$$ which is divisible by 9. ✓ base + induction (0.75p)

So, by mathematical induction we have shown that $4^n+15n-1$ is divisible by 9 for all $n \in \mathbb{N}$.

all steps correct + conclusion (1p)

3. Let a be the last digit of your student number

3.1 Write the complex numbers $z_1 = 1+(a+1)i$ and $z_2 = -\frac{1}{2}+(a+2)i$ in polar form. [0.5 pt]

Sol: for z_1: $r_1 = \sqrt{1+(a+1)^2} = \sqrt{1+a^2+2a+1} = \sqrt{a^2+2a+2}$ correct: $+(0.05p)$

$$\theta_1 = \arctan\left(\frac{a+1}{1}\right) = \arctan(a+1)$$ correct: $+(0.1p)$

$$\rightarrow \boxed{z_1 = r_1(\cos\theta_1 + i\sin\theta_2)}$$

for z_2: $r_2 = \sqrt{\frac{1}{4}+(a+2)^2} = \sqrt{\frac{1}{4}+a^2+4a+4} = \sqrt{a^2+4a+\frac{17}{4}}$ correct: $+(0.05p)$

$$\theta_2 = \arctan\left(-2(a+2)\right)$$ correct: $+(0.1p)$

$$\rightarrow \boxed{z_2 = r_2(\cos\theta_2 + i\sin\theta_2)}$$

$+(0.2p)$ if location in the Argand plane is correct. $\begin{cases} z_1 \text{ in the 1st quadrant.} \\ z_2 \text{ in the 2nd quadrant.} \end{cases}$

3.2 $z_1^2 z_2^3 = r_1^2\left(\cos(2\theta_1)+i\sin(2\theta_1)\right)\cdot r_2^3\left(\cos(3\theta_2)+i\sin(3\theta_2)\right)$

[0.5 pt] $= r_1^2 r_2^3\left(\cos(2\theta_1+3\theta_2)+i\sin(2\theta_1+3\theta_2)\right)$ (0.5 p) ; if this expression appears but numerical values are wrong → (0.3p)

3.3 $\dfrac{z_1^3}{z_2^4} = \dfrac{r_1^3(\cos(3\theta_1)+i\sin(3\theta_1))}{r_2^4(\cos(4\theta_2)+i\sin(4\theta_2))} = \dfrac{r_1^3}{r_2^4}\left[\cos(3\theta_1-4\theta_2)+i\sin(3\theta_1-4\theta_2)\right]$ (0.5 p)

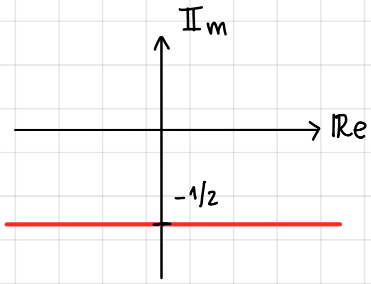
[0.5 pt]

; if this expression appears but numerical values are wrong → (0.3p)

4 (v_1) Sketch the solutions of $|z-2i| = |z+3i|$ [1.5 p]

Sol: $z = a+bi \rightarrow |a+(b-2)i| = |a+(b+3)i| \rightarrow \cancel{a^2} + (b-2)^2 = \cancel{a^2} + (b+3)^2 \rightarrow \cancel{b^2} - 4b + 4 = \cancel{b^2} + 6b + 9$

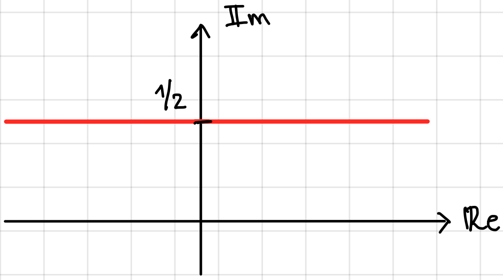
$\rightarrow 10b = -5 \rightarrow \boxed{b = -\frac{1}{2}}$



- criterion $b = -\frac{1}{2}$ (1 p)
- sketch (0.5 p)

4 (v_2) Sketch the solutions of $|z+2i| = |z-3i|$ [1.5 p]

Sol: $z = a+bi \rightarrow |a+(b+2)i| = |a+(b-3)i| \rightarrow \cancel{a^2} + (b+2)^2 = \cancel{a^2} + (b-3)^2 \rightarrow \cancel{b^2} + 4b + 4 = \cancel{b^2} - 6b + 9 \rightarrow 10b = 5 \rightarrow \boxed{b = \frac{1}{2}}$



- criterion $b = \frac{1}{2}$ (1.0 p)
- sketch (0.5 p)

5 (v_1) Prove using the ε,δ-definition of limit that $\lim_{x\to -1} (3x+1) = -2$. [1p]

Sol: preliminary analysis: $|3x+1+2| < \varepsilon \leftrightarrow |3x+3| < \varepsilon \leftrightarrow 3|x+1| < \varepsilon \leftrightarrow |x-(-1)| < \frac{\varepsilon}{3}$

$\rightarrow \delta = \frac{\varepsilon}{3}$ +0.25p

Proof: for any $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{3}$. Then

$0 < |x-(-1)| < \delta \rightarrow |x+1| < \delta \rightarrow |3x+3| < 3\delta \rightarrow |3x+1-(-2)| < 3\delta = \varepsilon$.

+0.75p

it follows from the ε,δ-definition of limit that $\lim_{x\to -1} (3x+1) = -2$.

5 (v_2) Prove using the ε,δ-definition of limit that $\lim_{x\to -1} (4x+2) = -2$ [1p]

Sol: preliminary analysis: $|4x+2-(-2)| < \varepsilon \leftrightarrow |4x+4| < \varepsilon \leftrightarrow 4|x-(-1)| < \varepsilon$

$\rightarrow \delta = \frac{\varepsilon}{4}$ +0.25p

Proof: for any $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{4}$. Then

$0 < |x-(-1)| < \delta \rightarrow |x+1| < \delta \rightarrow |4x+4| < 4\delta \rightarrow |4x+2-(-2)| < 4\delta = \varepsilon$.

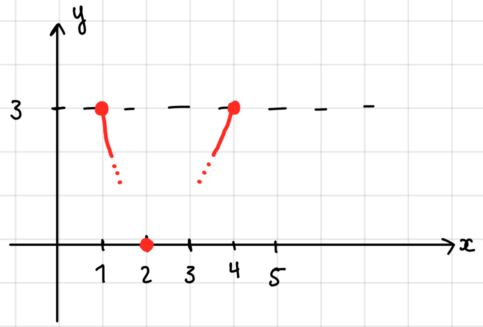
It follows from the ε,δ-definition of limit that $\lim_{x\to -1} (4x+2) = -2$

+0.75p

6 (v_1) Suppose that the function $f(x)$ is continuous on $[1,5]$ and that the only solutions of $f(x)=3$ are: $x=1$ and $x=4$. If $f(2)=0$, explain why $f(3)<3$.

[1.5 p]

Sol: sketch (optional but very useful).



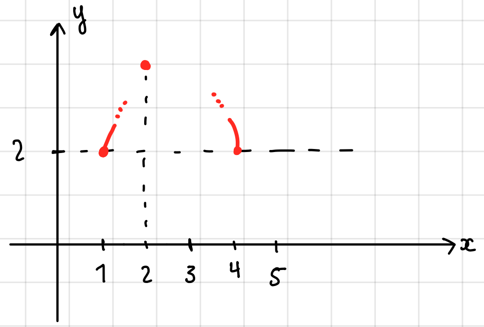
Suppose $f(3)\geq 3$. Since $f(x)$ is continuos, by the IVT, there is a $c\in(2,3)$: $f(c)=3$, or $f(3)=3$. Both lead to a contradiction, therefore $f(3)<3$.

- Any correct and complete explanation/proof (1.5 p)
- Any correct and logic, but incomplete, attempt (0.75 p)
- Any ilogic or incorrect attempt (0 p)

6 (v_2) Suppose that the function $f(x)$ is continuous on $[1,5]$ and that the only solutions of $f(x)=2$ are: $x=1$ and $x=4$. If $f(2)=4$, explain why $f(3)>2$.

[1.5 p]

Sol: sketch (optional but very useful).



Suppose $f(3)\leq 2$. Since $f(x)$ is continuos, by the IVT, there is a $c\in(2,3)$: $f(c)=2$, or $f(3)=2$. Both lead to a contradiction, therefore $f(3)>2$.

- Any correct and complete explanation/proof (1.5 p)
- Any correct and logic, but incomplete, attempt (0.75 p)
- Any ilogic or incorrect attempt (0 p)

7. Let n be all the digits of your student number. Find the n-th derivative of $f(x) = x e^{-x}$. [1.5 p]

Sol: ① We prove first that for any $k \in \mathbb{N}$, $f^{(k)}(x) = (-1)^k (x-k) e^{-x}$.

1: for $k=1$ we have $f'(x) = -(x-1)e^{-x}$ which is indeed correct.

2: assume $f^{(k)}(x) = (-1)^k (x-k) e^{-x}$. Then:

$$f^{(k+1)}(x) = \frac{d}{dx}\left[f^{(k)}(x)\right] = \frac{d}{dx}\left[(-1)^k (x-k)e^{-x}\right] = (-1)^k\left[e^{-x} - (x-k)e^{-x}\right]$$

$$= (-1)^k\left[-(x-(k+1))e^{-x}\right] = (-1)^{k+1}\left[x-(k+1)e^{-x}\right]$$

Thus, by mathematical induction if $f(x) = xe^{-x}$, then $f^{(k)}(x) = (-1)^k (x-k)e^{-x}$ $\forall k \in \mathbb{N}$.

any (not necessarily via induction)

- Proof + correct substitution of student number (1.5p)
- No proof, but good justification of the formula (e.g. $f', f'', f''', \ldots$) + correct substitution of student number (1p)
- Simply using the formula with no justification (0p)